Instructional Strategies *for* Mathematics in the Early Grades

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Introduction

This document is intended for program and curriculum experts interested in implementing evidence-based early grade mathematics programs. It was developed by the authors of this document, who are mathematics teaching and learning experts with extensive experience adapting evidence-based practices in low and middle-income contexts. Our collective field and research experience, combined with the existing evidence base, led us to focus on four instructional strategies that are key to effective mathematics instruction:

1. Respecting developmental progressions
2. Using mathematical models to represent abstract notions
3. Encouraging children to explain and justify their thinking
4. Making explicit connections for children between formal and informal math

While these four instructional strategies are very important, they are not the only instructional strategies that can result in improved learning outcomings. Effective early grade mathematics teachers draw from an extensive repertoire of evidence-based instructional strategies and strive to create a learning environment the supports that development of positive mathematical identities.

Structure of the document

The first part of this document provides an overview of each of these strategies. We begin by defining each strategy and explaining why it is important in the early grade mathematics classroom. We then look at how the strategy has or can be used or implemented in low and middle income classrooms, drawing on our extensive field experience.

The second part of the document presents ideas for introducing teachers to each of the instructional strategies, beginning with ideas for building teachers' knowledge of the strategy and supporting them as they begin to integrate it in their classroom. We then present suggestions to help teachers delve deeper and finally to expand and refine their understanding of the strategy.

The third part of the document examines cross-cutting issues that can have a bearing on the quality of the teaching and learning environment in mathematics classes, and the effectiveness of instructional strategies used.

Finally, the last part of the document contains two reference lists. The first outlines the references quoted in the document. The second, entitled Additional Resources, lists resources we have found helpful when developing early grade mathematics programs.

Citation:
Sitabkhan, Y., Davis, J., Earnest, D., Evans, N., Ketterlin-Geller, L., Lutfeali, S., Ngware, M., Perry, L., Pinto, C., Platas, L., Ralaingita, W., Smith, K., & Srikantaiah, D. (2019). Instructional Strategies for Mathematics in the Early Grades. A Working Paper Developed by the Mathematics Working Group.

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Instructional Strategies

Instructional Strategy 1: Developmental Progressions

What do we mean by developmental progressions?

We define developmental progressions in mathematics as research-based learning trajectories within and across mathematical domains that reflect increasingly sophisticated understanding. In other words, developmental progressions describe how children's learning progresses from simple to more complex understanding within a single domain (for instance, number) or subdomain (for instance, cardinality). Development progressions can be thought of as pathways that describe the key steps in children's understanding of concept. They are based on decades of cognitive research on how children learn. Teachers use this knowledge of how children are likely to progress to inform their teaching.[1]

An example of a developmental progression pathway for enumeration (counting a set of objects) is illustrated here [2, 3, 4]:

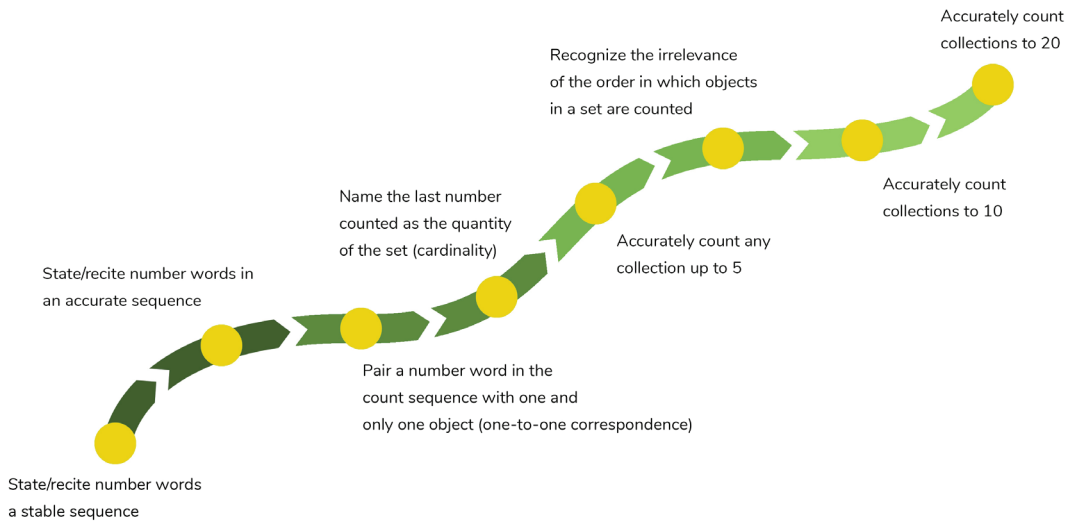

Figure 1: *Developmental progression pathway, Enumeration*

As in other developmental progressions that involve quantity, set size makes a difference in enumeration. For instance, children may be able to count a set of five objects with one-to-one correspondence, but may not have one-to-one correspondence for larger sets (such as 10) until they have more experience with the number word sequence and counting larger sets. Likewise, children may be able to name the cardinal value of a set of three objects due to the ability to subitize (instantly identify the quantity of a small set) before they can use one-to-one correspondence for that same set.

Why is it important?

Children come to classrooms with a wide variety of mathematical skills, rendering a one-size-fits-all approach inadequate to support all learners. The use of a developmental progressions thinking-framework in the classroom results in knowing where children are in their development, understanding where they came from, and how to support them in the future. This framework supports meeting children where they are, guided by content standards and research on children's learning, and allows teachers to sequence learning activities based on how children's thinking about a particular concept develops over time. Sequencing instruction this way ensures that children are presented with cognitively appropriate learning activities resulting in more effective learning.[2] Without knowledge of developmental progressions, teachers may fail to identify which skills children have mastered, are developing, or have not yet mastered, making it difficult to make instructional decisions with regard to tasks or activities that might further develop.

How can it be used in the classroom?

Determining where children are in specific developmental progressions (such as enumeration), requires teachers to consider children's prior knowledge. Prior knowledge is developmentally less complex, and instruction should both build upon and enrich these prior understandings, resulting in more sophisticated mathematical thinking. The progress pathway metaphor is particularly powerful for helping teachers understand that previously-learned content on the pathway facilitates the understanding of subsequent content. Teachers who come from a developmental progressions thinking-framework plan instruction according to developmental learning progressions. They continually watch and listen to identify how children are moving along those developmental progressions and adjust their teaching appropriately.

Teachers also need to understand how knowledge in one mathematical domain supports development in another domain. An example of this is the use of multiplication when determining the area of rectangles. Teachers need to keep in mind developmental progressions in several mathematical domains (in this case, operations and geometry) in the planning and execution of instruction.

Learning happens within contexts. While children move along a developmental progression path, context may affect the speed and depth at which they understand and make connections between concepts. Children may understand a mathematical concept when a particular model is used but fail to understand when another model is used. An example of this is the use of an area model compared to the use of a number line with regard to fraction learning (see figure 2 below). Children may gain understanding of fractions in one of these more easily. Additionally, understanding may be fragile, seeming to disappear from one class to the next. Children's problem-solving strategies also evolve, becoming more sophisticated as the problem demands increase. Addressing all of these issues effectively requires multiple presentations of a mathematical idea in multiple formats.

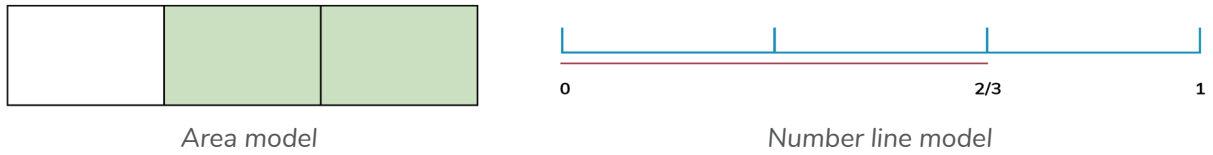

Figure 2: *Area versus number line model of the fraction 2/3*

It is important to note that conceptual understanding and numerical/arithmetic fluency are both important in the support of related developmental progressions. Research shows that children initially rely on conceptual and procedural strategies, but then gradually utilize memory retrieval. The resulting fluency allows for more cognitive resources to be applied to the more sophisticated pieces of complex mathematical problems. This means that teachers should be prepared to provide children with opportunities to engage in both types of strategies to support their movement along a developmental progressions path.

Ultimately, teachers must use their knowledge of where children are in their developmental progression to differentiate instruction. Ideally, this would mean that the teacher has knowledge of each and every child's place in each mathematical subdomain progression. However, in large classes, this may not be possible. Gathering information in order to understand at which levels most children are may be a workable solution to this problem. Teachers would conduct formative assessments to ascertain children's understandings, and then provide instruction that meets this variation in needs in the classroom. The use of peer instruction may be helpful in these instances, strengthening the skills of both advanced and less-advanced children.

Instructional Strategy 2: Mathematical Models

What do we mean by mathematical models?

A model in mathematics refers to any picture, drawing, or object that represents targeted mathematical ideas. When mathematical models are physical objects, such as counters, they are commonly referred to as manipulatives.[5] A number line, which is a drawing, is also a mathematical model. Too often, people treat mathematical models as something extra for struggling children. This is far from true—all children benefit from using mathematical models. That is because mathematical models enable children to "see" abstract mathematical concepts. They allow children to reason concretely with mathematical ideas. When working with mathematical models, the symbols we typically associate with mathematics become meaningful. This enables children to grapple with the logic and meaning underlying the symbols, procedures and rules.

What counts as a mathematical model?

Examples of common manipulatives include base-10 blocks, pattern blocks, geometric figures, dice, fraction pieces (see *figure 3 below*).

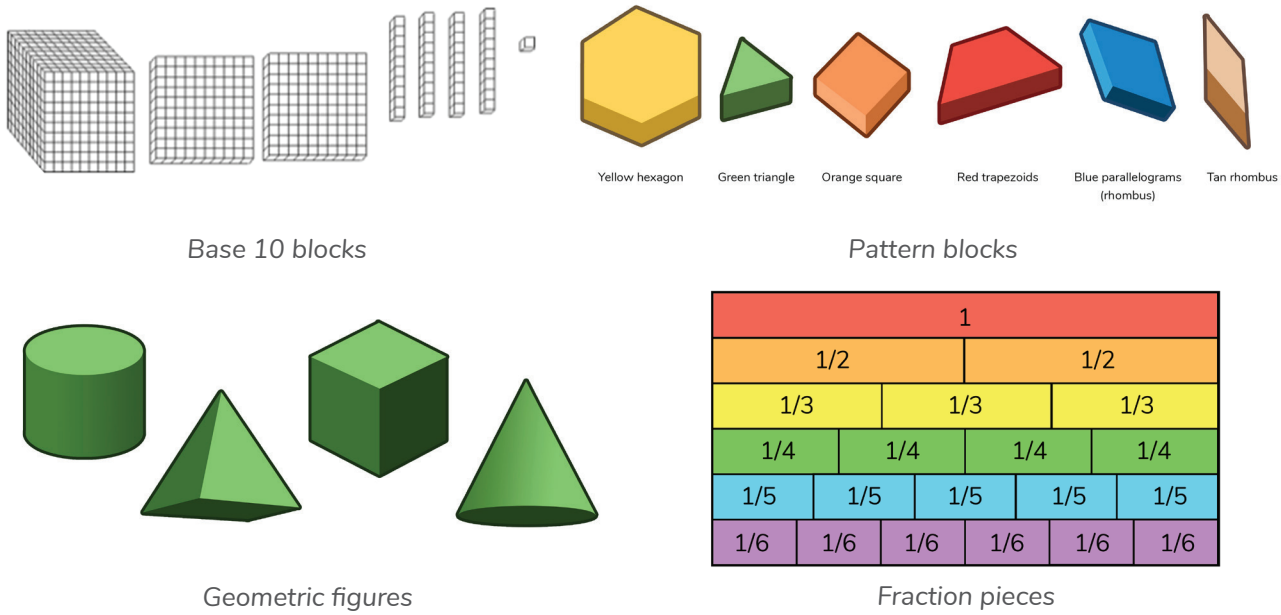

Figure 3: Common manipulatives

Manipulatives model specific mathematical ideas. As children progress across grades, they tend to work first with concrete manipulatives that they can hold, move, and put together as they reason about early mathematical ideas, followed by drawn or pictorial representations. Finally, they will increasingly rely on abstract mathematical symbols to represent mathematical ideas or concepts.

Let's consider base-10 blocks, which are materials that feature a unit (a small cube, representing one), a ten rod (a rod made up of 10 small cubes, representing 10), and a hundred block (a square made up of 10 rods, representing 100). Often, the unit blocks may be improvised, using a stick as the unit or one and a bundle of 10 sticks to represent 10 (*See Figure 4*). These concrete materials were designed to make visible key properties of the base-10 number system. For example, they render visible the idea of exchange that underlies computation. Consider the problem 22 - 7. We can start with two ten bundles of 10 to represent 20 and two sticks to represent two units. (*See Figure 5*).

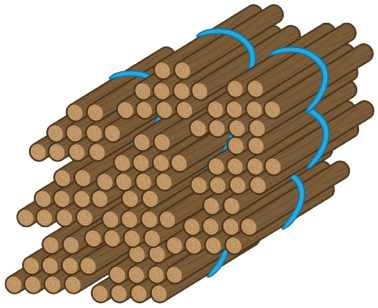

Figure 4: *An illustration of a bundle of 100 sticks bundled in groups of ten*

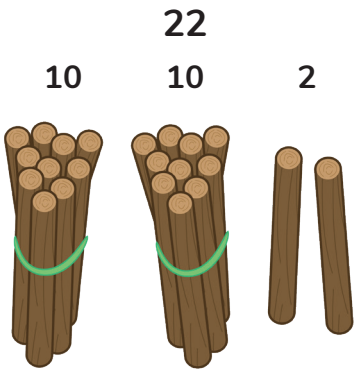

Figure 5: *Two bundles of 10, and two units*

Because we cannot take seven units away from two units we must exchange one of the bundles of 10 for 10 units, thereby leaving us with one bundle of ten and 12 sticks or units (*Figure* 6). Because the manipulatives highlight the relationship between ten units and one group of ten, this exchange models what is commonly referred to as "borrowing" in the subtraction computation algorithm.

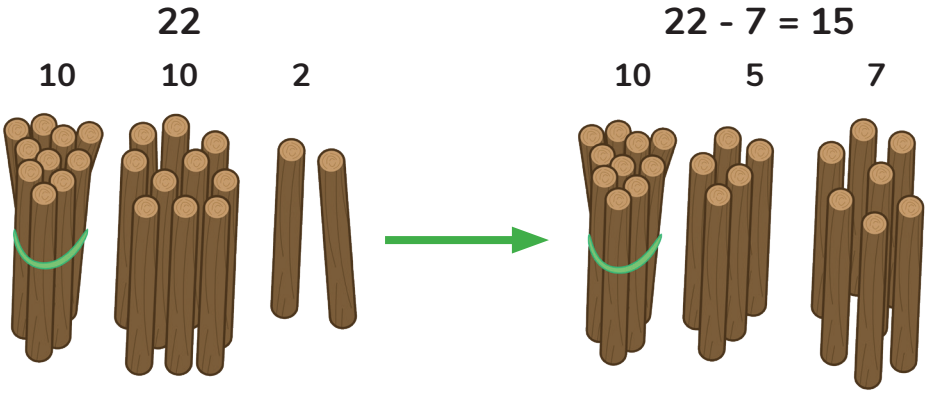

Figure 6 (left): *22 with one ten exchanged for ten ones, which then allows a child to model (right) 22 -7 = 15 using base-10 properties.*

As mentioned previously, mathematical models can also be visual or drawn. Examples of this include a number line (*Figure* 8), a ten frame (also known as a 2x5 grid) (*Figure* 7), area models, pictures, and graphs. Visual, drawn, or pictorial models even include the symbols we commonly associate with mathematics (e.g., =, +, -), as well as written numerals. The number line (*Figure* 8) is a printed representation that models, or shows, order and magnitude. In the figure, we can see that the distances between whole numbers remain constant, such that the distance between 1 and 2 is the same as between 3 and 4.

Textbox 1: How to use models in classroom instruction

- Define the math content of a lesson
- Map the math features of model onto the content
- Plan the lesson and the models to use
- Plan for the management of the models

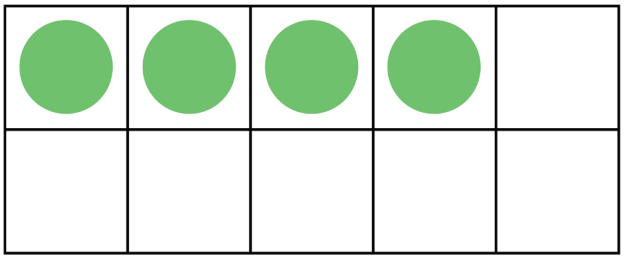

Figure 7: *A ten frame showing "4"*

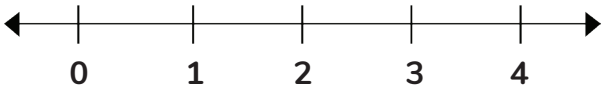

Figure 8: *A number line model*

Why are they important?

In general, young children tend to think first in concrete ways before thinking in more abstract ways. This pattern is similar in learning mathematics. To learn mathematics with meaning, young children need to begin with concrete materials that model target mathematical ideas[6]. Children then tend to work with pictorial or drawn models. Eventually, the goal is to support children in being able to reason with abstract symbols without needing a physical mathematical model of those symbols. For example, children first need opportunities to count objects and associate a number, such as 4, with concrete objects, such as four bottle tops. Then, they can move to pictorial or drawn models, such as using a ten-frame to show the quantity of "4", as seen in Figure 7. Through these activities, the symbol "4" comes to represent a quantity of four objects; eventually, children can reason with 4 without associating it with a count of objects.

How can models be used in the classroom?

Mathematical models are essential in learning mathematics with meaning. However, the mere presence of mathematical models in the classroom does not guarantee improved learning[7]. Teachers must carefully choose which model to use and know how that model highlights important mathematical ideas with which children are engaging[8].

To support teachers in using models strategically, we encourage teachers to **define the math content of a lesson**, and, based on this, identify a model (physical or drawn) that will allow children to explore these mathematical ideas. To identify a model, it is helpful to **map the math features of the model onto the content**. For example, if exploring geometry, cut outs of shapes are important because of how they model mathematical ideas related to shapes, sides, angles, and similar shapes. The teacher may also draw upon shapes already in the classroom, such as the window and door frames, shapes of furniture, and other objects.

To begin to use mathematical models, teachers must begin to **plan the lesson and the models to use**, including a **plan for the management of the models** if they use physical manipulatives. It can be productive and beneficial to make such decisions prior to starting a lesson, particularly as physical materials often need to be distributed among children. In addition, research shows that providing children with a high level of guidance on how to use models properly leads to better learning outcomes[9].

Consider the example below. To teach children about combinations from 3 to 10, a teacher can use ten frames with two-colored counters to highlight different ways of composing or decomposing a number. The ten-frame here features four green counters and one yellow counter, thereby modeling one combination to make 5, i.e., 4 + 1 (*Figure 9*). Teachers can use the ten frame in Figure 7 above, with four green counters, to model, "How many more do you need in order to make 10?" The target mathematics in this case is combinations to make ten, with the ten frame highlighting these ideas and providing a way for children to see the mathematics that eventually will be symbolized as 10 = 4 + 6.

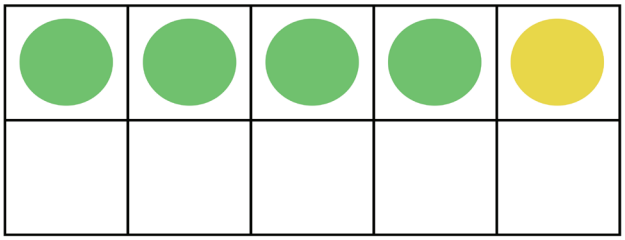

Figure 9: *A ten frame showing "5"*

Instructional Strategy 3: Explanation and Justification

What do we mean by encouraging children to explain or justify their thinking or their solutions?

Children explain or justify their thinking when they describe how they arrived at a solution to a problem or comment on or critique the thinking, arguments or solutions of others. Explaining and justifying includes inviting children to make predictions, analyze mathematical situations, explore or propose other possible solutions or present arguments for a particular solution. Children can explain or justify their thinking orally, in writing, or by using drawings, diagrams or manipulatives.

Why is it important?

Having children explain or justify their thinking is important because it:

- *Deepens children's understanding*. When children explain their thinking and justify their solutions - either to the teacher or to each other – they organize and clarify concepts in their own minds. They can also become aware of and correct their misunderstandings, fill in gaps in their knowledge, and acquire new strategies.[10,11,12,13]

- *Increases children's motivation*. When teachers ask children to explain and justify their thinking, teachers communicate that they respect and value children's thinking[14]. This can lead to children developing more positive view of mathematics and of their own abilities as mathematics learners[15,16] . These positive dispositions can in turn further support learning[17].

- ***Allows teachers to tailor their instruction according to the level of children's understanding***. When teachers listen to children's thinking, they are able to identify what children know as well as the gaps in their understanding and misconceptions that need to be addressed. This information allows teachers to deliver more effective instruction[18,19].

How can it be used in the classroom?

The six instructional practices described below can encourage children's explanation and justification:

- ***Teacher modeling of explanation and justification*** – When teachers say things like "I think that the answer will be larger than 10 because both of the numbers we are adding are bigger than 5, and 5 is half of 10," they provide children with a concrete model of what explaining and justifying sound like.

- ***Asking questions that encourage explanation and justification.*** The simple questions in textbox 2 can be used in any lesson to foster explanation and justification.

Textbox 2: Five Simple Questions that Encourage Explanation and Justification

How did you get that answer?

How do you know?

What do you notice about…?

How is this the same or different from…?

How else could you have… (solved that problem, figured out what the word means…)?

- ***Viewing children's errors as a learning opportunity*** - Errors represent a window into children's misunderstandings and hence are a valuable diagnostic and remediation tool[18, 20]. Questions like those in textbox 2 can help teachers uncover what children have misunderstood. When teachers treat errors as learning opportunities, the stigma attached to being wrong is reduced and children come to realize that mistakes are part of the learning process[21]. Teachers who recognize common misconceptions and know how to best address them in the classroom are more effective[22,23].

- ***Providing sufficient "wait or think time" after asking a question to allow all children to think deeply about the answer***[24,25]. Simple directions like "Can you put your hands down and give everyone a minute to think?" can encourage children to think more deeply about a concept or question.

- ***Having children solve problems in pairs or in small groups***. Group or partner arrangements enhance engagement and provide a way for children to exchange and test ideas – which generates a higher level of thinking[26].

- ***Ensuring that children understand and use correct math vocabulary***. Mathematical words such as "less than", "more", "half" can have different meaning in mathematical versus every day conversations. Teachers can help children understand and use mathematical terms correctly by modeling the correct use of terms, requiring the children use terms when explaining, and pointing out the special meaning some words have in mathematics.

Overcoming challenges to encouraging the use of explanation and justification in the classroom

The following three challenges can impede or discourage the use of explanation and justification in early primary classrooms:

1) ***Lack of teacher acess to the types of rich mathematical tasks that promote explanation and justification.*** Traditional textbooks may not contain the types of problems or activities that provide meaningful explanation and justification activities (see textbox 3).

   Solution: Curriculum can be supplemented with rich tasks and procedural-type problems can be transformed into open-ended problems. For example, 6 + 7 = 13, can be replaced by the problem "*The answer is 13. What might the question or the problem be?*" or "*Shirin has 5 apples. Christabel has 3. Who has more?*" can easily become "*Shirin has 5 apples. Christabel has 3. What can you say about the situation?*"

Textbox 3: Nature of rich tasks

Tasks that foster explanation and justification generally have multiple solutions (i.e., are open-ended) and/or can be solved in multiple ways, are interesting for the children and deal with important or core mathematical ideas.

2) ***The presence of a classroom culture that does not place a great deal of value on having children explain and justify their thinking***. Shifting some responsibility for learning from the teacher to the child can be challenging in contexts where the teacher is expected to be the source of knowledge. Engaging children in explanation and justification activities may also result in classrooms that have a higher noise level than traditional classrooms, or in classroom appear to be less orderly[29,30], leading to the perception that little learning is taking place. As well, allocating valuable learning time to explanation and justification-type activities may be a source of tension for teachers who feel pressured to "cover the curriculum." Finally, standardized assessments often do not emphasize – and hence value - explanation and justification. This can discourage teachers from allocating class time to developing children's ability to explain and justify their thinking.

Solution: Encouraging teachers to view children's errors as learning opportunities can be a catalyst for shifting interaction patterns in the classroom. Children may also need explicit discussions to convince them that behaviors that may be inappropriate outside the classroom (questioning adults, for example) are appropriate and even desirable in math classes. Teacher supervisors who understand the value of explanation and justification-type practices can create a culture that accepts and expects these practices to take place in the classroom behaviors.

3) ***Limited teacher mathematical knowledge.*** Teachers' ability to ask probing, open-ended questions and manage ensuing conversations requires strong conceptual understanding and experience managing children's responses in constructive ways[30,31,33]. This in turn may make them reluctant to ask "why" and "how" questions in their own classrooms[32.]

Solution: Teachers' conceptual understanding can be developed through targeted teacher training programs[31.] Teachers can also be taught strategies for dealing with situations where they do not know or are unsure of the answer, for example responding with "*That's a good question. Let's note it on our list of good questions and see if anyone can come up with an answer or explanation this week.*"

Instructional Strategy 4: Connections Between Formal and Informal Mathematics

What are connections between formal and informal mathematics?

A key instructional strategy in early mathematics classrooms is creating connections between children's formal and informal mathematics.

Humans are born with an innate sense of quantity, such as differences in quantities of small sets and an approximate number line, that is refined and developed as children interact with their world [34,35,36]. These interactions allow children to build on their innate sense of quantity to develop informal mathematics. Children go about their day and encounter problems to be solved, such as figuring out how many people are in a room, how far away something is, or how much the food costs in a market[37,38,39,40,41,42,43,44,45]. They create strategies to solve the everyday problem they are facing at that moment. Informal mathematics, then, is specific to a situation and often involves solving problems orally where an understanding of symbols such as the equals sign is not needed to find a solution. For example, a child may be purchasing something for 250 shillings, and needs to make sure s/he receives the correct amount of change from a 500-shillings note. The strategy that the child uses for this transaction is particular to the problem in front of them, and does not necessarily lead to a generalized understanding of arithmetic.

Formal mathematics is the mathematical that is learned in school through formal instruction. Formal mathematics is symbolic, using written numerals, written symbols, and other mathematical models, such as an area model and number line. Because formal mathematics uses symbols and models, it can be generalized to a new problem. For example, once children understand the algorithm for subtraction with two digits numbers involving regrouping, they can extend this algorithm to a new problem, such as subtraction with three-digit numbers.

Key differences between formal and informal mathematics:

Formal Mathematics	Informal Mathematics
Learned in school through instruction	Learned through everyday experiences
Symbolic	Often non-symbolic
Often written	Often oral
Can be generalized	Not easily generalized
Example: an algorithm to solve the problem 500 - 250, which, in this case, involves borrowing	Example: A strategy used to figure out how much change you will receive from a 500-shilling note for a purchase of 250 shillings.

Why is it important?

Although formal and informal mathematics often involve similar types of problems, they differ in terms of the context. In everyday life, problems that need to be solved are authentic and closely tied to a situation. In school, children are often asked to solve problems that may be devoid of meaning. While we want to support abstract aspects of mathematics, everyday mathematical situations are a critical component of one's overall understanding that needs to be included in school instruction.

Making connections between formal and informal mathematics is critical in the early years. Connections provide a way for curriculum developers and teachers to validate and recognize that children come to school with mathematical knowledge. When first attending school, whether at the primary or pre-primary level, children already are engaged in solving math problems. Knowing, recognizing, and building upon the knowledge they have developed is key. In addition, this recognition helps make all mathematics relevant to children's lives, which can increase motivation and interest.

Making connections between formal and informal math supports the development of strong foundations in early mathematical concepts. Many foundational concepts in the early years, such as number sense, simple operations, and fractions, are concepts that arise frequently in children's everyday activities. For example, young children often solve fair-share problems, such as equally dividing a number of sweets among their siblings. The knowledge and strategies they have developed through solving fair share problems can add meaning to the concept of division when it is introduced in school.

How can it be used in the classroom?

Two strategies can support teachers in building connections between formal and informal math. The first is the strategy of **bridging**, or teacher-led discussions where teachers explicitly connect a problem in class with an out-of-school problem[46]. Consider the problem above of 500 - 250. When a teacher links the problem presented in class to the act of receiving change in an economic transaction, it creates a bridge--the everyday act of buying something gives meaning to the abstract problem presented in class, and the strategy learned in class can be applied to other problems children may encounter in their everyday experiences.

Second, the strategy of **using problems that are connected to everyday mathematics** can lend itself to connection-making. One has to be careful that the word problems are connected to the mathematics children do in their everyday lives and not presenting an application of a concept[47,48]. For example, consider a typical word problem that one may encounter in a textbook: *There are 8 children on a bus. 5 are boys, and the rest are girls. How many girls are on the bus?* On first glance, it may appear that this problem is connected to the mathematics in children's lives, as many children may take a bus to school. However, in order to be connected, it must be a problem that children routinely solve as part of their everyday life. In this case, unless children regularly calculate how many girls are on the bus, it is not connected to the mathematics children do, and therefore would be considered an application of a concept. If word problems are connected to the math in children's lives, they are a valuable method to help children connect their informal and formal knowledge.

Photo by: Jeff Holt/Save the Children

How can teachers get started?

Using the four key instructional strategies can be a daunting task. Below, we provide exemplars of how to support teachers to understand and implement the strategy in their classrooms. Note that the suggestions are only examples, and not meant to be an exhaustive or prescriptive.

Suggestions are provided for four different entry points in teachers' professional development pathways. Depending on teachers' prior skill levels and prior knowledge, program implementers may choose to focus on a single entry point, or target several points at once.

Developmental Progressions

Building teacher knowledge	Read and gain an understanding of what a developmental progression in a mathematical sub-domain (such as addition) looks like for children who are younger, the same age, and older than the children in your classroom
Beginning to integrate use	Complete a formative assessment task that assesses knowledge within the developmental progressions described above. Keep it simple. For instance, ask a few children to come up to the board to solve an addition problem and explain, or model explanation of, how they got their answer. Follow up with simple probes as needed. Or, review homework on addition to ascertain where children are succeeding and struggling.
Delving deeper	Create, and engage children in, an addition problem that is slightly more difficult than those the children are generally succeeding at. Watch to see how they solve the problem. Are they retrieving the answers from memory? Counting on their fingers? Where do they seem to have understanding and where are they still struggling?
Expanding and Refining	Use different set sizes (or addends) – are they more successful at some than others? If children are struggling, try changing the problem to make the mathematics more obvious.

Mathematical Models

Building teacher knowledge	Become familiar with the use of the model needed for a lesson, including the properties of the model that lend themselves to solving the problem. For example, if using a 10 frame, practice drawing one quickly and efficiently. Ask: • How does this model help children understand the concept? • What struggles may children have using this model?
Beginning to integrate use	Accurately use the model during whole class or small group instruction during a lesson. Ensure explanations and demonstrations are correct, so that they are accessible to all children. Engage children in using the model through questions, such as: • Can you show me the number “6” using this number line or number chart? • How can you show “6” using these counters?
Delving deeper	Support all children to use model in meaningful ways by ensuring that the models are in the hands of the children. Plan and make sure there are enough models so that each child has a model. During a lesson, ask children to use the model to show what they are learning.
Expanding and Refining	Think about how a particular model (for example, a number line) can be used to model other concepts (such as addition, or fractions). For example, if using a number line to show the number "6", ask: • How do I use this number line to show 6 +1? Think about the targeted concept for a lesson, and identify what other models can support children’s learning of that concept. For example, if using counters to compare quantities, use a number line to extend children’s learning.

Explanation and Justification

Building teacher knowledge	Describe the rationale for asking children to explain and justify their thinking when learning mathematics. Tell children that it is okay to make mistakes in the mathematics classroom and describe how mistakes will be managed in the classroom. Using correct mathematical vocabulary, model how to explain and justify thinking for a question like 8 + 6 = ___. Practice explaining how to solve questions or explain your thinking.
Beginning to integrate use	When children give an answer, whether it is right or wrong, ask one of two questions: • How did you get that answer? • How do you know? Listen to children's thinking in order to point out where their reasoning went wrong. Corrective feedback is important in the learning process. Wait 3 to 5 seconds after asking a question before selecting the child to answer, in order to give everyone a chance to think deeply.
Delving deeper	Plan or find mathematical tasks or problems that lend themselves to rich explanation and justification. Flip around traditional textbook problems in the textbook to make them richer (see examples in Explanation and Justification section). Have children work in pairs on problems so that they talk about solutions and explain their thinking, using quiet voices. Provide children with the list of questions in Textbox in the Explanation and Justification section to discuss during partner work.
Expanding and Refining	When children are working on rich tasks or problems, ask simple, but context appropriate questions like those in the textbox in the Explanation and Justification section to encourage children to share their thinking or their answers. Insist that children use correct mathematical vocabulary in their responses.

Connecting Formal and Informal Mathematics

Building teacher knowledge	Choose a content area (ex. simple operations, number sense, geometry), and brainstorm how children use the related mathematical concepts in their everyday lives. For example, with single digit addition, a child may combine numbers in game play, at the market, and doing chores at home.
Beginning to integrate use	Ask children to share their experiences using questions such as: • Can anyone tell me about a time you went to the market? How many pieces of each kind? How did you know how many you had altogether?
Delving deeper	Link the math that children say they use outside of school with the math they are learning in school. For example, show how buying pieces of fruit in the market and knowing how many there are altogether is the same as adding two numbers together in math class.
Expanding and Refining	Create stories based on the math that children use outside of school and use these stories to introduce new concepts.

Photo by: Jeff Holt/Save the Children

Cross Cutting Issues

There are several cross-cutting issues that that can have a bearing on the quality of the teaching and learning environment in mathematics classes, and the effectiveness of the instructional strategies outlined in the briefs.

Delivery of Instruction

Teachers can optimally structure learning activities using different combinations of exploration, modeling, guided learning, and practicing independently, in pairs, and in small groups. Combinations may differ according to lesson objective, content focus, and the context (such as class size).

Role of Language

Learning in second language: Because many children learn mathematics in a second or third language, teachers can support children in developing conceptual understanding by talking colloquially in a home language to increase understanding (e.g., code switching). Teachers can encourage children to use a home language when discussing concepts.

Mathematical vocabulary: Children need to develop an understanding of mathematical terms if they are to make sense of mathematical ideas. In order to support "math talk," teachers can model and foster the use and understanding of accurate mathematical terms and expressions.

Classroom Climate

To support and encourage children's engagement, teachers can create a classroom climate that fosters positive math identities by encouraging children to meaningfully engage and persevere through challenging tasks, including those that are outside of the children's comfort zone.

Assessment

Teachers can improve their teaching and support children's learning by developing effective assessment practices that balance the use of summative assessment for evaluation purposes and formative assessment practices that can provide them with meaningful information about children's learning.

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